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ON EQUILIBRIUM INSTABILITY IN HOLONOMIC MECHANICAL SYSTEMS WITH PARTIAL DISSIPATION

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We prove a theorem on the instability of the equilibrium of a dissipative system in the absence of a maximum of the force function. The dissipation is partial and is absent only in one of the degrees of freedom. The proof is based not on the linearization of the differential equations but on Liapunov's direct method and uses a somewhat modified form of Krasovskii's theorem. The instability is established for systems with arbitrary nonlinear dissipative forces and an isolated equilibrium.

1. Statement of the problem. Let $q' = (q_1, q_2, \ldots, q_n)$ be the generalized coordinates of a holonomic mechanical system with n degrees of freedom (here and later the prime denotes transposition). We assume that the kinetic energy is a quadratic form in the generalized velocities $q_1^{\boldsymbol\cdot}, q_2^{\boldsymbol\cdot}, \ldots, q_n^{\boldsymbol\cdot}$

$$2T(q, q^{\boldsymbol\cdot}) = \sum_{i,j=1}^{n} a_{ij}(q) q_i^{\boldsymbol\cdot} q_j^{\boldsymbol\cdot} = (q^{\boldsymbol\cdot})' A(q) q^{\boldsymbol\cdot} \qquad (1.1)$$

where $A(q) = \| a_{ij} \|$. We assume that the functions $a_{ij}(q)$ are continuously differentiable in some neighborhood of the point $q = 0$, the matrix A is symmetric, and quadratic form (1.1) is positive definite in $q^{\boldsymbol\cdot}$.

Let the force function $U(q)$ also be continuously differentiable and, besides conservative forces, let there act only dissipative forces, so that the equations of motion have the form

$$d \,/\, dt\, (\partial L \,/\, \partial q_i^{\cdot}) - \partial L \,/\, \partial q_i = Q_i\, (q,\, q^{\cdot}) \tag{1.2}$$

Here

$$L\,(q,\, q^{\cdot}) = T\,(q,\, q^{\cdot}) + U\,(q)$$

moreover

$$D(q,\, q^{\cdot}) = \sum_{i=1}^{n} Q_i(q,\, q^{\cdot})\, q_i^{\cdot} \leqslant 0 \tag{1.3}$$

for all values of $(q,\, q^{\cdot})$ in some neighborhood of $(q = 0,\, q^{\cdot} = 0)$.

The generalized dissipative forces $Q_i\,(q,\, q^{\cdot}),\ i = 1,\, 2,\, \ldots,\, n,$ are continuous in q and $q^{\cdot}$ and $Q_i\,(q,\, 0) = 0,\ i = 1,\, 2,\, \ldots,\, n,$ for all values of q. Since T is positive definite, the equations of motion can always be reduced to the standard form

$$q^{\cdot\cdot} = f\,(q,\, q^{\cdot}) \tag{1.4}$$

Each solution of system (1.4) corresponds to a possible motion. We assume that the system (1.4) being considered satisfies the conditions for the existence (for $t \geqslant 0$) and the uniqueness of the solution, and we denote the solution corresponding to the initial values $q\,(t = 0) = q_0$ and $q^{\cdot}\,(t = 0) = q_0^{\cdot}$, by $q\,(q_0,\, q_0^{\cdot},\, t)$. We assume, further, that the system has an equilibrium position at $q = 0$, so that

$$\partial U \,/\, \partial q_i = 0 \quad \text{for } q = 0,\ i = 1, 2, \ldots, n \tag{1.5}$$

We consider the question of the Liapunov stability of this trivial solution. Without loss of generality we assume that $U\,(0) = 0$.

If at the point $q = 0$ the function U has a strict maximum (strict in the sense defined by Bourbaki [1]: "maximum relatif strict"), then it immediately follows from Liapunov's theorem that the equilibrium position is stable. Indeed, choosing $T - U$ as the Liapunov function, by virtue of system (1.2), we have

$$d(T - U)\,/\,dt = \sum_{i=1}^{n} Q_i(q,\, q^{\cdot})\, q_i^{\cdot}$$

that is what shows the stability of the solution. When $D\,(q,\, q^{\cdot}) \equiv 0$ we have the Lagrange-Dirichlet theorem. If $D\,(q,\, q^{\cdot})$ is negative definite in $q^{\cdot}$ and the equilibrium position is isolated, i.e. if some neighborhood of the point $q = 0$ exists in which there are no other equilibrium positions, then the equilibrium is asymptotically stable. If the force function $U\,(q)$ does not have a maximum at $q = 0$, then in dissipation-free systems $(D\,(q,\, q^{\cdot}) = 0)$ the equilibrium is apparently unstable in all cases of practical interest. However, at the level of strict theoretical investigations additional conditions must be imposed on the function $U\,(q)$. This is shown by the very well-known counterexample of Painlevé [2].

The mathematical difficulties arising in the proof of such instability theorems are quite considerable in cases of both dissipative as well as nondissipative systems (see [2 – 6]). If the equilibrium position $q = 0$ is isolated and if in any arbitrarily small neighborhood of $q = 0$ there are points at which U takes positive values, then only when the function $D\,(q,\, q^{\cdot})$ is negative definite in $q^{\cdot}$, a trivial solution of this problem is known: the equilibrium is unstable. This can be derived from a modification of Liapunov's theorem (for example, see Theorem 15.1 in [7]). On the other hand, let us assume that the dissipation is partial. In this case it is still not known whether the absence, in the above defined sense, of a maximum of $U\,(q)$ implies that $q = 0$ is necessarily unstable. In principle we need to keep in mind the possibility of the stabilization of a previously unstable equilibrium by the dissipative forces. This question is extremely difficult and no general results are known to date. Below we prove instability for the isolated equilibrium case if dissipation is defined on $n-1$ generalized coordinates and if in any arbitrarily small neighborhood of $q=0$ we can find points q^i such that $U(q^i)>0$. We say that the dissipation is reduced to $n-1$ coordinate if $Q_n\equiv 0$, this is assumed below. We need the following Theorem A on the instability of the trivial solution $x=0$ of the differential equation $x^{\cdot}=X(x)$.

Theorem A. Let the following conditions be fulfilled:

1°. The function $V(x)$ is defined and continuously differentiable in

$$G=\{x \mid \|x\|\leqslant H\},\qquad V(0)=0,\ H>0$$

2°. $dV/dt\geqslant 0$ in G along the trajectories of the equation $x^{\cdot}=X(x)$.

3°. A sequence $x^1, x^2, \ldots, V(x^i)>0,\ i=1,2,\ldots,\ x^i\to 0$ as $i\to\infty$ and a certain number H_1, $H>H_1>0$, exist such that not even one of the solutions $x(x^i,t)$ contains a semitrajectory starting in $G_1=\{x \mid \|x\|\leqslant H_1\}$, along which $dV/dt\equiv 0$. Then $x=0$ is unstable.

We say that a solution $x(x^\circ,t)$ contains a semitrajectory starting in G_1, along which $dV/dt\equiv 0$, if some $t_*\geqslant 0$ exists such that the solution lies wholly in G_1 for $0\leqslant t\leqslant t_*$ and if for $t\geqslant t_*$ we have $dV/dt\equiv 0$ for all $x(x^\circ,t_{**})$ such that $x(x^\circ,t)\in G_1$, $t_*\leqslant t\leqslant t_{**}$. This theorem is a direct corollary of Krasovskii's Theorem 15.1 in [7], where it is required that no semitrajectories exist along which $dV/dt\equiv 0$. However, such semitrajectories can be allowed if they start outside the specified fixed neighborhood of $x=0$. This does not alter the proof.

2. Instability theorem for systems with partial dissipation. We give the precise statement of the instability theorem announced above.

Theorem. Let a holonomic dissipative mechanical system with n degrees of freedom have an equilibrium at $q=0$ and let it satisfy the following conditions:

1°. The equilibrium is isolated ;

2°. In an arbitrarily small neighborhood of $q=0$ there are points q^i such that $U(q^i)>0$ $(U(0)=0)$;

3°. The function

$$D(q,q^{\cdot})=\sum_{j=1}^{n-1}Q_j(q,q^{\cdot})q^{\cdot}_j$$

is negative definite in $q_1^{\cdot}, q_2^{\cdot}, \ldots, \dot q_{n-1}$ for all q from some neighborhood of $q=0$, while $Q_n(q,q^{\cdot})\equiv 0$;

4°. The coefficients a_{sn} $(s=1,2,\ldots,n-1)$ occurring in the expression for the kinetic energy do not depend on q_n, while a_{nn} is a constant. Then the equilibrium is unstable.

Proof. The equilibrium $q=0$ is isolated, therefore, a number $H^*>0$ exists such that $q=0$ is the unique equilibrium position of the system in $G^*=\{(q,q^{\cdot}) \mid \|q\|+\|q^{\cdot}\|<H^*\}$. To prove the instability theorem we need two lemmas, corresponding to two types of sequences of the form

$$(q^i,(q^{\cdot})^i)\in G^*,\qquad i=1,2,\ldots\qquad (q^i,(q^{\cdot})^i)\to(0,0)\quad \text{as}\quad i\to\infty$$

$$T(q^i,(q^{\cdot})^i)-U(q^i)<0$$

Such sequences can always be found if the theorem's hypotheses are fulfilled.

Type 1. We call $(q^i, (q^\cdot)^i)$, $i = 1, 2, \ldots$, a sequence of Type 1 if a certain number H_1^*, $H^* > H_1^* > 0$ exists such that not even one of the motions $q(q^i, (q^\cdot)^i, t)$, $q^\cdot(q^i, (q^\cdot)^i, t)$ (the solution of (1.4)) contains a semitrajectory starting in $G_1^* = \{(q, q^\cdot) \mid \|q\| + \|q^\cdot\| < H_1^*\}$, along which $D(q, q^\cdot) \equiv 0$.

Lemma 1. If all the hypotheses of the theorem are fulfilled for some mechnical system and if a sequence of Type 1 exists, then the equilibrium $q = 0$ is unstable.

This follows immediately from Theorem A. Having set $V = -T + U$, we have $V^\cdot = -D$, and all of the requirements of Theorem A are fulfilled.

Type 2. We call $(q^i, (q^\cdot)^i)$ a sequence of Type 2 if a number H_2^*, $H^* > H_2^* > 0$, exists such that all motions $q(q^i, (q^\cdot)^i, t)$, $q^\cdot(q^i, (q^\cdot)^i, t)$ contain semitrajectories starting in $G_2^* = \{(q, q^\cdot) \mid \|q\| + \|q^\cdot\| \leqslant H_2^*\}$, along which $D(q, q^\cdot) \equiv 0$.

Lemma 2. If all the hypotheses of the theorem are fulfilled for some mechanical system and if a sequence of Type 2 exists, then the equilibrium $q = 0$ is unstable.

To prove this we examine one of the motions $q(q^i, (q^\cdot)^i, t)$ for an arbitrary i. Let us show that for each i the phase point reaches the boundary of region G_2^* in finite time. If $D \equiv 0$, then $q_1^\cdot = q_2^\cdot = \ldots = q_{n-1} = 0$, $q_1^i = c_1^i$, $q_2^i = c^i{}_2, \ldots$, $q_{n-1}^i = c_{n-1}^i$ ($c_s^i = \text{const}$, $s = 1, 2, \ldots, n-1$). Since the first $n-1$ coordinates are constant, the motion exists only in one degree of freedom. (For this case, as will be shown, we can exhibit instability without particular difficulty).

By t_i we denote the magnitude of t such that $D(q(q^i, (q^\cdot)^i, t), q^\cdot(q^i, (q^\cdot)^i, t)) \equiv 0$ for $t \geqslant t_i$. The point $(q(q^i, (q^\cdot)^i, t), q^\cdot(q^i, (q^\cdot)^i, t))$ belongs to G_2^*. We show that the algebraic equation

$$\frac{\partial}{\partial q_n} U(c_1^i, c_2^i, \ldots, c_{n-1}^i, q_n) = 0 \tag{2.1}$$

has no solution in G_2^* on the trajectory being considered $(t \geqslant t_i)$. We consider the equations of motion

$$\sum_{s=1}^{n} a_{rs} q_s^{\cdot\cdot} + \frac{1}{2} \sum_{u=1}^{n} \sum_{v=1}^{n} \left(\frac{\partial a_{ru}}{\partial q_v} + \frac{\partial a_{rv}}{\partial q_u} - \frac{\partial a_{uv}}{\partial q_r} \right) q_u^\cdot q_v^\cdot - \frac{\partial U}{\partial q_r} = 0$$

$$r = 1, 2, \ldots, n$$

which follow from (1.2) when $Q_r = 0$. From $q_s = c_s^i$, $s = 1, 2, \ldots, n-1$, and from assumption 4° of the instability theorem (assumption 4° of the instability theorem can be replaced by the weakened condition $2\partial a_{rn} / \partial q_n = \partial a_{nn} / \partial q_r$, $r = 1, 2, \ldots, n$), we have

$$a_{rn} q_n^{\cdot\cdot} - \partial U / \partial q_r = 0, \qquad r = 1, 2, \ldots, n \tag{2.2}$$

If $\partial U / \partial q_n = 0$ for some value of q_n, then $q_n^{\cdot\cdot} = 0$ also, since $a_{nn} > 0$. Then from the first $n-1$ equations of (2.2) it follows that $\partial U / \partial q_r = 0$ ($r = 1, 2, \ldots, n-1$) at this same point. This is impossible at any point of G_2^*, besides the center, since the equilibrium is isolated. However, the phase point cannot approach the origin because $h_i = T - U < 0$. This signifies that Eq. (2.1) does not have solutions on the trajectory being considered.

The motion $q(q^i, (q^\cdot)^i, t)$ takes place only in one degree of freedom (corresponding to q_n), therefore, it should belong to one of the following types:

1°. $(q(t),\ q^\cdot(t)) \in G_2^*, \quad t \geqslant t_i$

$$(g(t),\ g^\cdot(t)) \to (g^*,\ 0) \in G_2^* \quad \text{as } t \to \infty$$

$$g'(t) = (c_1^i,\ c_2^i, \ldots, c_{n-1}^i,\ q_n(t)),\ (g^\cdot(t))' = (0, 0, \ldots, 0,\ q_n^\cdot(t))$$

$$(g^*)' = (c_1^i,\ c_2^i, \ldots, c_{n-1}^i,\ q_n^*)$$

2°. $(q(t),\ q^\cdot(t)) \in G_2^*$, $t \geqslant t_i$, and $q_n(q^i, (q^\cdot)^i, t)$ is periodic ;

3°. $(q(t),\ q^\cdot(t))$ reaches the boundary of G_2^* in finite time.

There is no such simple classification for motions in more than one degree of freedom.

In Case 1° the equilibrium position would be at the point $(g^*, 0) \in G_2^*$, which follows from the continuity of the right-hand sides of (1.4). This equilibrium would be different from $q = 0,\ q^\cdot = 0$, since $h_i < 0$. Consequently, Case 1° can be excluded (by assumption a unique equilibrium point exists at the origin in G_2^*).

In Case 2° the function $q_n(t)$ varies in the closed interval $[g_1,\ g_2]$. The function $|\partial U / \partial q_n|$ is continuous in q_n and, consequently, achieves a minimum on this interval. From what was stated above we know that this minimum does not equal zero. Consequently, a number $a_i > 0$ exists such that $|\partial U / \partial q_n| > a_i$ on the motion being considered, i.e. the function $U(c_1^i, c_2^i, \ldots, c_{n-1}^i, q_n)$ is strictly monotone in q_n, and the equation $h_i = -U(q_n)$ has no more than one solution in $[g_1,\ g_2]$. Consequently, there is no more than one point with zero velocity in the motion $q(q^i, (q^\cdot)^i, t)$ for $t \geqslant t_i$. This contradicts the assumption of periodicity of the solution, and Case 2° also can be excluded.

Consequently, only Case 3° is possible. Thus, the sequence $(q^i, (q^\cdot)^i) \to (0, 0)$, $i = 1, 2, \ldots$ exists and, moreover, each solution $(q(q^i, (q^\cdot)^i, t),\ q^\cdot(q^i, (q^\cdot)^i, t))$ reaches the boundary of region G_2^* in finite time. The equilibrium is unstable and the proof of Lemma 2 is completed.

The proof of the instability theorem is now obvious. Let us consider an arbitrary sequence $(q^i, (q^\cdot)^i) \in G^*$, $i = 1, 2, \ldots$, satisfying the conditions

$$(q^i, (q^\cdot)^i) \to (0, 0) \quad \text{as } i \to \infty$$
$$T(q^i, (q^\cdot)^i) - U(q^i) < 0, \quad i = 1, 2, \ldots$$

From this sequence let us try to pick out a subsequence of Type 1. If this is possible, then the instability follows from Lemma 1. If this is impossible, then there should exist a certain p such that for $i > p$ all terms of the sequence correspond to motions which contain semitrajectories starting in G^*, along which $D(q, q^\cdot) \equiv 0$. This signifies that a sequence of Type 2 exists. Then the instability is guaranteed by Lemma 2, and the proof of the instability theorem is completed.

One requirement of the instability theorem can be slightly weakened. It is not obligatory to require that $Q_n(q, q^\cdot) = 0$ for all $(q, q^\cdot) \in G^*$, but it suffices to require that Q_n equal zero in $G^{**} \subset G^*$, where

$$G^{**} = G^* \cap \{(q, q^\cdot) \mid T(q, q^\cdot) - U(q) \leqslant 0\}$$

This is obvious since in the theorem's proof we considered only the motions with the initial conditions $(q^i, (q^\cdot)^i) \in G^{**}$ under which the phase points can reach the boundary of G^{**} only at points belonging also to the boundary of G^*.

3. Some unanswered questions. In the instability theorem proposed here we have dealt exclusively with systems with dissipation absent only in one of the degrees of freedom. Motions on which energy dissipation does not take place and which possibly exist can then be treated as the solutions of the equations of motion of a system with one degree of freedom. On the other hand, in the case of a conservative system with one degree of freedom it can be shown that each motion with a negative mechanical energy $T - U < 0$ leaves some neighborhood of an isolated equilibrium position. This still has not been proved for systems with several degrees of freedom. If such a proof were to be obtained, we could generalize the proposed theorem to the case of systems with arbitrary partial dissipation.

Chetaev [4] proved that all motions with zero initial values of the momenta leave a neighborhood of the equilibrium position. However, this is insufficient for the generalization of the proposed instability theorem, because such an assertion should concern all motions starting in G^{**}. Possibly we can succeed in deriving this property by means of geometrical reasoning. If the problem is formulated on the basis of the Jacobi's principle, then the problem reduces to the question of the behavior of geodesics in a pseudo-Riemannian space. The answer can apparently be found using the existence theorems of the calculus of variations (for example, see[5]).

The author conjectures that the following generalization of the proposed instability theorem is valid. Let a discrete holonomic autonomous dissipative ($D\ (q,\ q^{\cdot})$ is constantly negative) system have an isolated equilibrium position at $q = 0$. In an arbitrarily small neighborhood of the equilibrium there exist points q^i at which $U\ (q^i) > 0$, then the equilibrium is unstable ($U\ (0) = 0$). The conditions that the equilibrium position is isolated and that U can take positive values are essential.

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